

# Hamiltonian Systems on Complex Grassmann Manifold. Holonomy and Schrödinger Equation

Zakaria Giunashvili

Department of Theoretical Physics,  
Institute of Mathematics, Georgian Academy of Sciences  
Tbilisi, Georgia

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## Abstract

Differential geometric structures such as the principal bundle for the canonical vector bundle on a complex Grassmann manifold, the canonical connection form on this bundle, the canonical symplectic form on a complex Grassmann manifold and the corresponding dynamical systems are investigated. The Grassmann manifold is considered as an orbit of the co-adjoint action and the symplectic form is described as the restriction of the canonical Poisson structure on a Lie coalgebra. The holonomy of the connection on the principal bundle over Grassmannian and its relation with **Berry phase** is considered and investigated for the integral curves of Hamiltonian dynamical systems.

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# 1 Introduction

The main purposes of this paper are

1. to review all the *canonical* differential geometric structures on Grassmann manifold of the finite-dimensional complex subspaces of a complex Hilbert space.
2. to investigate the canonical symplectic structure, the corresponding dynamical systems and the holonomies of the integral curves of these dynamical systems for complex Grassmann manifolds.

We start from the differential structure on the complex Grassmann manifold and then investigate the connection on the canonical principal bundle and its holonomies. It turns out that the canonical connection on this principal bundle and its holonomy plays an important role in the theory of the geometric quantum computations (see [3], [4], [5], [10], [11], [12]), which indicates the universality of the Grassmann manifold not only in the theory of fiber bundles, but also in the geometric theory of the quantum computations. The central point in these topics is the pure geometric fact, that any unitary transformation of the fiber of the canonical fiber bundle over a complex Grassmann manifold can be obtained as the holonomy of a closed curve. But for physical reasons, we need not any closed curve on a Grassmann manifold, but only "physical" ones; i.e., the curves that are the integral curves for a Hamiltonian dynamical system. For an integral curve of a Hamiltonian system on a Grassmann manifold, the corresponding curve on the total space of the canonical bundle is the solution of the corresponding Schrödinger equation. But the latter, in general, is not a horizontal curve, which is the lifting the curve on the base manifold. It "becomes" horizontal in **adiabatic limit**. We investigate the geometry of such curves and its relation with Berry phase.

## 2 The Differential Structure and Coordinate Systems on Complex Grassmann Manifold

Let  $\mathcal{H}$  be a complex Hilbert space, infinite or finite -dimensional, as required. For any positive integer number  $m$  let us denote by  $Gr_m(\mathcal{H})$  the set of all  $m$ -dimensional complex subspaces of the space  $\mathcal{H}$ . It is clear that any  $m$ -dimensional subspace  $X$  of the Hilbert space  $\mathcal{H}$  can be uniquely defined by

the corresponding operator of orthogonal projection  $\mathcal{P}(X) : \mathcal{H} \longrightarrow X \subset \mathcal{H}$ . The operator  $\mathcal{P}(X)$  is characterised by the properties:  $\mathcal{P}(X)^* = \mathcal{P}(X)$  and  $\text{trace}(\mathcal{P}(X)) = \dim(X)$ .

Let us denote by  $\mathcal{P}_m(\mathcal{H})$  the set

$$\mathcal{P}_m(\mathcal{H}) = \{P : \mathcal{H} \longrightarrow \mathcal{H} \mid P^2 = P, P^* = P, \text{trace}(P) = m\} \quad (1)$$

We have that the mapping  $\mathcal{P} : Gr_m(\mathcal{H}) \longrightarrow \mathcal{P}_m(\mathcal{H})$  is a bijection (see [4], [5], [6]) and defines an injection of the set  $Gr_m(\mathcal{H})$  in the vector space of Hermitian operators on the Hilbert space  $\mathcal{H}$ . This injection induces a topology and a differential structure on the set  $Gr_m(\mathcal{H})$ . The set  $Gr_m(\mathcal{H})$ , together with this differential structure is known as the Grassmann manifold of  $m$ -dimensional complex subspaces of the Hilbert space  $\mathcal{H}$ . As it follows from the above discussion, further we can identify the following two objects:  $\mathcal{P}_m(\mathcal{H})$  and  $Gr_m(\mathcal{H})$ .

Any fixed element  $X \in Gr_m(\mathcal{H})$  defines a mapping

$$\Gamma_X : Hom(X, X^\perp) \longrightarrow Gr_m(\mathcal{H})$$

$$\text{where, for } \varphi \in Hom(X, X^\perp) : \Gamma_X(\varphi) = \{x + \varphi(x) \mid x \in X\}$$

In other words, the subspace corresponding to the linear mapping  $X \xrightarrow{\varphi} X^\perp$  is the graph of the mapping  $\varphi$ . In some cases, we shall omit the subscript  $X$  in the expression  $\Gamma_X$  and write just  $\Gamma$ . This mapping is an injection of the vector space  $Hom(X, X^\perp)$  into the set  $Gr_m(\mathcal{H})$  and its image is the subset

$$\{Y \in Gr_m(\mathcal{H}) \mid Y \cap X^\perp = \{0\}\} \equiv \mathcal{W}_X$$

It follows that varying the element  $X \in Gr_m(\mathcal{H})$  we can cover the Grassmann manifold by the open subsets  $\mathcal{W}_X$ . Actually, it is sufficient to take only a finite number of the elements  $X \in Gr_m(\mathcal{H})$  to cover the total manifold by the sets of the type  $\mathcal{W}_X$ .

Hence, we can state that a pair  $(Hom(X, X^\perp), \Gamma_X)$ , for  $X \in Gr_m(\mathcal{H})$ , defines a local coordinate system in the neighborhood of the point  $X$  (see [8]).

Now, let us find the expression for the diffeomorphism

$$\mathcal{P} : Gr_m(\mathcal{H}) \longrightarrow \mathcal{P}_m(\mathcal{H})$$

in the above local coordinate system. In other words, the problem is to find the coordinate expression for the the following composition map

$$\Phi_X : Hom(X, X^\perp) \xrightarrow{\Gamma_X} Gr_m(\mathcal{H}) \xrightarrow{\mathcal{P}} \mathcal{P}_m(\mathcal{H})$$

For any  $f \in Hom(X, X^\perp)$  the graph of the operator  $-f^* \in Hom(X^*, X)$  is the orthogonal subspace of the space  $\Gamma_X(f)$ . Indeed, for  $x \in X$  and  $u \in X^\perp$  we have the following

$$\begin{aligned} \langle x + f(x), u - f^*(u) \rangle &= \underbrace{\langle x, u \rangle}_0 - \langle x, f^*(u) \rangle + \langle f(x), u \rangle - \underbrace{\langle f(x), f^*(u) \rangle}_0 = \\ &= \langle f(x), u \rangle - \langle x, f^*(u) \rangle = 0 \text{ (by the definition of dual operator).} \end{aligned}$$

For  $f \in Hom(X, X^\perp)$  consider the following operator on the Hilbert space  $\mathcal{H}$ :

$$\tilde{f} = \begin{bmatrix} \mathbf{1} & -f^* \\ f & \mathbf{1} \end{bmatrix} : X \oplus X^\perp \longrightarrow X \oplus X^\perp$$

This operator is automorphism and it is easy to verify that  $\tilde{f}(X) = \Gamma(f)$  and  $\tilde{f}(X^\perp) = \Gamma(-f^*) = \Gamma(f)^\perp$ . It is clear that if  $P_U$  is the projection operator on a subspace  $U \subset W$  corresponding to a decomposition of a vector space  $W = U \oplus V$  and  $A : W \longrightarrow W$  is an automorphism, then the projection operator on the subspace  $A(U)$ , corresponding to the decomposition  $W = A(U) \oplus A(V)$ , is  $P_{A(U)} = AP_U A^{-1}$ . Therefore, the projection operator, corresponding to the decomposition  $\mathcal{H} = \Gamma(f) \oplus \Gamma(f)^\perp = \tilde{f}(X) \oplus \tilde{f}(X^\perp)$  is  $\mathcal{P}(\Gamma(f)) = \tilde{f}\mathcal{P}(X)\tilde{f}^{-1}$ . The explicit expression for the operator  $\tilde{f}^{-1}$ , for the decomposition  $\mathcal{H} = X \oplus X^\perp$  is

$$\begin{bmatrix} \mathbf{1} & -f^* \\ f & \mathbf{1} \end{bmatrix}^{-1} = \begin{bmatrix} (1 + f^*f)^{-1} & f^*(1 + ff^*)^{-1} \\ -f(1 + f^*f)^{-1} & (1 + ff^*)^{-1} \end{bmatrix}$$

**Remark 1** *The operator  $1 + f^*f : X \longrightarrow X$  is invertible, because*

$$\begin{aligned} (1 + f^*f)(x) = 0 &\Rightarrow (f^*f)(x) = -x \Rightarrow \langle (f^*f)(x), x \rangle = -\|x\|^2 \\ &\Rightarrow \|f(x)\|^2 = -\|x\|^2 \Rightarrow x = 0 \end{aligned}$$

*The same is true for the operator  $1 + ff^* : X^\perp \longrightarrow X^\perp$ .*

Therefore, the explicit expression for the projection operator  $\mathcal{P}(\Gamma(f))$ , corresponding to the decomposition  $\mathcal{H} = X \oplus X^\perp$  is

$$\begin{aligned} \mathcal{P}(\Gamma(f)) &= \begin{bmatrix} \mathbf{1} & -f^* \\ f & \mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} (1 + f^*f)^{-1} & f^*(1 + ff^*)^{-1} \\ -f(1 + f^*f)^{-1} & (1 + ff^*)^{-1} \end{bmatrix} = \\ &= \begin{bmatrix} (1 + f^*f)^{-1} & f^*(1 + ff^*)^{-1} \\ f(1 + f^*f)^{-1} & ff^*(1 + ff^*)^{-1} \end{bmatrix} \end{aligned} \quad (2)$$

The action of the group  $U(\mathcal{H})$  (the group of the unitary transformations of the Hilbert space  $\mathcal{H}$ , on  $\mathcal{H}$ ) induces the action of this group on the Grassmann manifold. This action is transitive and for any point  $X \in Gr_m(\mathcal{H})$ , the corresponding stabilizer subgroup is  $U(X) \times U(X^\perp)$ . Therefore, the manifold  $Gr_m(\mathcal{H})$  can be considered as the homogeneous space (see, for example, [4], [5], [6])

$$Gr_m(\mathcal{H}) \cong \frac{U(\mathcal{H})}{U(X) \times U(X^\perp)}$$

For any unitary transformation  $u : \mathcal{H} \longrightarrow \mathcal{H}$ , the corresponding diffeomorphism of the Grassmann manifold  $gr(u) : Gr_m(\mathcal{H}) \longrightarrow Gr_m(\mathcal{H})$  induces a transformation of the local coordinate systems

$$\tilde{u} : Hom(X, X^\perp) \longrightarrow Hom(Y, Y^\perp)$$

where  $Y = u(X)$  (and therefore  $Y^\perp = u(X^\perp)$ ). This transformation is defined by the condition:  $\Gamma(\tilde{u}(f)) = u(\Gamma(f))$ . We have that

$$u(\Gamma(f)) = \{ux + uf(x) \mid x \in X\} = \{y + uf(u^{-1}y) \mid y \in u(X) \equiv Y\}$$

which implies that  $u(\Gamma(f)) = \Gamma(ufu^{-1})$ . Hence, the transformation of the local coordinate system is of the form

$$\begin{aligned} \tilde{u} : Hom(X, X^\perp) &\longrightarrow Hom(u(X), u(X^\perp)) \\ \tilde{u}(f) &= ufu^{-1} \end{aligned} \quad (3)$$

The tangent space of the manifold  $Gr_m(\mathcal{H})$  at a point  $X \in Gr_m(\mathcal{H})$  can be identified with the vector space  $Hom(X, X^\perp)$  (see [8]). The identification of the manifold  $Gr_m(\mathcal{H})$  with the manifold of projections  $\mathcal{P}_m(\mathcal{H})$ , gives another representation of the tangent space  $T_X(Gr_m(\mathcal{H}))$ , which follows from 1:

$$T_X(Gr_m(\mathcal{H})) \cong \{\Phi : \mathcal{H} \longrightarrow \mathcal{H} \mid \Phi \text{ is linear and } \mathcal{P}(X) \circ \Phi + \Phi \circ \mathcal{P}(X) = \Phi, \Phi^* = \Phi, \text{trace}(\Phi) = 0\} \quad (4)$$

This is exactly the set of such operators  $\Phi : \mathcal{H} \longrightarrow \mathcal{H}$ , that the decomposition of  $\Phi$ , corresponding to the decomposition of the Hilbert space  $\mathcal{H} = X \oplus X^\perp$  is of the form

$$\Phi = \begin{bmatrix} 0 & \varphi^* \\ \varphi & 0 \end{bmatrix}$$

The differential of the mapping  $f \mapsto \mathcal{P}(\Gamma(f))$  (see 2), at the point  $0 \in Hom(X, X^\perp)$ , gives the isomorphism between the two representations of the tangent space  $T_X(Gr_m(\mathcal{H}))$ :

$$Hom(X, X^\perp) \ni \varphi \mapsto \mathcal{P}'_X(\varphi) = \begin{bmatrix} 0 & \varphi^* \\ \varphi & 0 \end{bmatrix} \quad (5)$$

Recall that an action of a Lie group  $G$  on a smooth manifold  $M$ , gives rise of the homomorphism from the Lie algebra of  $G$  to the Lie algebra of vector fields on the manifold  $M$ : for  $v \in \mathfrak{g}$ , where  $\mathfrak{g}$  denotes the Lie algebra of the Lie group  $G$ , the corresponding vector field  $\tilde{v}$ , induced by the action of the group on  $M$ , is defined as (see [9])

$$\tilde{v}(x) = \frac{\partial F}{\partial G}(1, x)(v)$$

where  $F : G \times M \longrightarrow M$  is the mapping corresponding to the action of the group  $G$  on the manifold  $M$ .

So, the action of the group  $U(\mathcal{H})$  on the manifold  $Gr_m(\mathcal{H})$ , gives rise of the homomorphism from the Lie algebra of  $U(\mathcal{H})$ , to the Lie algebra of vector fields on the manifold  $Gr_m(\mathcal{H})$ . To describe this homomorphism, recall that if  $\mathcal{P}(X)$  is the orthogonal projection corresponding to the element  $X \in Gr_m(\mathcal{H})$ , then for any orthogonal transformation  $g \in U(\mathcal{H})$ , the orthogonal

projection corresponding to  $g(X)$  is  $\mathcal{P}(g(X)) = g\mathcal{P}(X)g^{-1}$ . Therefore, for any fixed point  $X \in Gr_m(\mathcal{H})$ , we have a mapping

$$U(\mathcal{H}) \ni g \mapsto g\mathcal{P}(X)g^{-1} \in \mathcal{P}_m(\mathcal{H}) \cong Gr_m(\mathcal{H})$$

This implies that the tangent vector corresponding to an element  $u \in \mathfrak{u}(\mathcal{H})$  of the Lie algebra of the Lie group  $U(\mathcal{H})$ , at a point  $X \in Gr_m(\mathcal{H})$  is  $\tilde{u}_X = [u, \mathcal{P}(X)]$ , where  $[\cdot, \cdot]$  denotes the commutator of two operators. If the decomposition of the linear operator  $u : \mathcal{H} \longrightarrow \mathcal{H}$ , corresponding to the decomposition  $\mathcal{H} = X \oplus X^\perp$  is

$$u = \begin{bmatrix} U_{XX} & -U_{XX^\perp}^* \\ U_{XX^\perp} & U_{X^\perp X^\perp} \end{bmatrix}$$

then it is clear that

$$\tilde{u}_X = [u, \mathcal{P}(X)] = \begin{bmatrix} 0 & U_{XX^\perp}^* \\ U_{XX^\perp} & 0 \end{bmatrix}$$

This, together with the formula 5, implies that the vector field corresponding to the Lie algebra element  $u \in \mathfrak{u}(\mathcal{H})$  via the action of the group  $U(\mathcal{H})$  on  $Gr_m(\mathcal{H})$  is

$$\tilde{u}_X = U_{XX^\perp} = (1 - \mathcal{P}(X)) \circ u \in Hom(X, X^\perp) \cong T_X(Gr_m(\mathcal{H})) \quad (6)$$

### 3 Some Operations Over Connections on Vector Bundles

In this section we shall review some facts from the theory of vector bundles and connections in the context of this work.

For any vector bundle  $\pi : E \longrightarrow M$ , we denote by  $S(E)$  the space of smooth sections of this vector bundle.

Let  $V \xrightarrow{p} M$  and  $W \xrightarrow{q} M$  be two vector bundles over a smooth manifold  $M$ . Let  $S(V) \xrightarrow{\nabla^V} S(T^*(M) \otimes V)$  and  $S(W) \xrightarrow{\nabla^W} S(T^*(M) \otimes W)$  be connections (covariant derivations) on them. We define the following operations over these connections:



**The direct (or Whitney) sum of two connections.** Define a connection  $\nabla = \nabla^V \oplus \nabla^W$  on the Whitney sum of the vector bundles  $(V, M, p)$  and  $(W, M, q)$  as follows: for a vector field  $\xi$  on the manifold  $M$  and a section  $s = s_1 \oplus s_2$  of the vector bundle  $V \oplus W$  let

$$\nabla_\xi(s) = \nabla_\xi^V(s_1) + \nabla_\xi^W(s_2) \quad (7)$$

**The tensor product of two connections.** Define a connection  $\nabla = \nabla^V \otimes \nabla^W$  on the tensor product of the vector bundles  $(V, M, p)$  and  $(W, M, q)$  as

$$\nabla_\xi(s_1 \otimes s_2) = \nabla_\xi^V(s_1) \otimes s_2 + s_1 \otimes \nabla_\xi^W(s_2) \quad (8)$$

**The connection on the dual vector bundle.** Define a connection  $\nabla^{V^*}$  on the vector bundle  $p_* : V^* \rightarrow M$  a fiber of which at a point  $x \in M$  is the dual vector space of the vector space  $p^{-1}(x)$

$$\xi \langle s, \tau \rangle = \langle \nabla_\xi^V(s), \tau \rangle + \langle s, \nabla_\xi^{V^*}(\tau) \rangle \quad (9)$$

where  $s$  is a section of the vector bundle  $V$  and  $\tau$  is a section of the dual vector bundle  $V^*$ , and  $\langle s, \tau \rangle$  is the function on the manifold  $M$  obtained by the pairing of the sections  $s$  and  $\tau$ .

**The connection on the bundle of homomorphisms.** Let

$$Hom(V, W) \xrightarrow{h} M$$

be the fiber bundle the fiber of which at a point  $x \in M$  is the space of all homomorphisms from the vector space  $p^{-1}(x)$  to the vector space  $q^{-1}(x)$ . In the case of finite dimensional fibers, the fiber bundle is canonically isomorphic to the fiber bundle  $h : V^* \otimes W \rightarrow M$ . We have defined the connections on the dual spaces bundle and the tensor products bundle, therefore, the covariant derivation  $\nabla = Hom(\nabla^V, \nabla^W) = \nabla^{V^*} \otimes \nabla^W$  on the space of sections  $S(Hom(V, W))$  can be defined as  $\nabla_\xi(\tau \otimes s) = \nabla_\xi^{V^*}(\tau) \otimes s + \tau \otimes \nabla_\xi^W(s)$  for  $\tau \in S(V^*)$  and  $s \in S(W)$ . This implies that for  $t \in S(V)$  we have the following

$$\begin{aligned} \nabla_\xi(\tau \otimes s)(t) &= \langle \nabla_\xi^{V^*}(\tau), t \rangle \cdot s + \tau(t) \cdot \nabla_\xi^W(s) = \\ &= \xi(\tau(t)) \cdot s + \tau(t) \cdot \nabla_\xi^W(s) - \langle \tau, \nabla_\xi^V(t) \rangle \cdot s = \\ &= \nabla_\xi^W(\tau(t) \cdot s) - \langle \tau, \nabla_\xi^V(t) \rangle \cdot s \end{aligned}$$

which suggests the following more elegant expression for the covariant derivation  $\nabla = \text{Hom}(\nabla^V, \nabla^W)$  on the space of sections  $S(\text{Hom}(V, W))$ :

$$\text{for } f \in S(\text{Hom}(V, W)) \text{ and } t \in S(V) \text{ let} \quad (10)$$

$$\nabla_\xi(f)(t) = \nabla_\xi^W(f(t)) - f(\nabla_\xi^V(t))$$

It is easy to verify that the results of all the above defined operations satisfy the conditions required for covariant derivations on vector bundles.

Let  $p : V \longrightarrow M$  and  $q : W \longrightarrow M$  be such vector bundles over a smooth manifold  $M$  that their Whitney sum is a trivial vector bundle:  $V \oplus W \cong M \times H$ . This situation gives rise of a connection on the vector bundles  $(V, M, p)$  and  $(W, M, q)$  as follows. Any section of the vector bundle  $V$  can be regarded as a function on  $M$  with values in the vector space  $H$ . Let us denote this function under  $\tilde{s} : M \longrightarrow H$ . For a vector field  $\xi$  on the manifold  $M$ , define the covariant derivation of the section  $s$  as  $\nabla_\xi(s)(x) = P_{V_x}(\tilde{s}_x'(\xi_x))$ ,  $\forall x \in M$ . Here  $P_{V_x}$  is the projection operator  $P_{V_x} : H \longrightarrow V_x$  corresponding to the decomposition  $H = V_x \oplus W_x$ . The covariant derivation on the sections of the vector bundle  $W$  can be defined analogically. It can be verified directly that the above defined operation satisfies the conditions required for covariant derivations.

**Remark 2** If  $V_1, V_2, W_1$  and  $W_2$  are vector bundles over the manifold  $M$  such that  $V_1 \oplus W_1 \cong M \times H_1$  and  $V_2 \oplus W_2 \cong M \times H_2$  and  $\nabla^1$  and  $\nabla^2$  are the connections on  $V_1$  and  $V_2$ , accordingly, as defined above, then the covariant derivation on the vector bundle  $V_1 \otimes V_2$  induced by the decomposition

$$(V_1 \otimes V_2) \oplus ((V_1 \otimes W_2) \oplus (W_1 \otimes V_2) \oplus (W_1 \otimes W_2)) \cong M \times (H_1 \otimes H_2)$$

is the same as  $\nabla = \nabla^1 \otimes \nabla^2$  (see the formula 8).

**Definition 1** Let  $\pi : T(M) \longrightarrow M$  be the tangent vector bundle for a smooth manifold  $M$  and  $p : V \longrightarrow M$  be such vector bundle that the Whitney sum  $T(M) \oplus V$  is trivial bundle  $M \times H$ . We call the decomposition of a trivial fiber bundle,  $M \times H = T(M) \oplus V$ , **integrable** if the induced covariant derivation  $\nabla$  on the sections of the tangent bundle (i.e., vector fields on  $M$ ) satisfies the condition

$$\nabla_X(Y) - \nabla_Y(X) = [X, Y] \quad (11)$$

for any two vector fields  $X$  and  $Y$  on the manifold  $M$ .

If the vector space  $H$ , which is the fiber of the Whitney sum, is a Hilbert space, and for each point  $x \in M$ , the vector space  $V_x$  (the fiber of the vector bundle  $V$  at the point  $x$ ) is the orthogonal complement of the subspace  $T_x(M) \subset H$ , then the covariant derivation  $\nabla$  satisfies the condition

$$X\langle Y, Z \rangle = \langle \nabla_X(Y), Z \rangle + \langle Y, \nabla_X(Z) \rangle$$

Hence, we can state that the connection  $\nabla$ , induced by the decomposition  $M \times H = T(M) \oplus V$ , is the covariant derivation corresponding to the Levi-Civita connection on the manifold  $M$ , for the metric induced by the embedding  $T(M) \subset M \times H$ .

**Remark 3** *If  $\Phi : M \longrightarrow H$  is a smooth mapping, such that for each point  $x \in M$  the linear mapping  $\Phi'_x : T_x(M) \longrightarrow H$  is a monomorphism, then the vector bundle  $p : \Phi^\perp(H) \longrightarrow M$ , where  $p^{-1}(x) = \text{Im}(\Phi'_x)^\perp$ , is such that the Whitney sum  $T(M) \oplus \Phi^\perp(H)$  is trivial:  $M \times H$ ; and the connection (covariant derivation) induced by the decomposition  $M \times H = T(M) \oplus \Phi^\perp(H)$  is the Levi-Civita connection on  $T(M)$ .*

Let  $\pi : P \longrightarrow M$  be a principal bundle with a structure group  $G$ , acting from right on the total space  $P$ . Let  $F$  be a vector space on which the Lie group  $G$  acts from left. The associated vector bundle over the manifold  $M$  is defined as the vector bundle with the total space  $(P \times V)/G \equiv P_V$ , where the quotient is taken under the right action of the group  $G$  on the manifold  $P \times V$ :  $(p, v) \mapsto (pg, g^{-1}v)$ , for each  $(p, v) \in P \times G$  and  $g \in G$  (see [9]). The projection mapping  $\pi^V : P_V \longrightarrow M$  for this vector bundle is defined as

$$\pi^V([p, v]) = \pi(p), \quad \forall [p, v] \in (P \times V)/G$$

It follows from the definition that any section of the associated vector bundle can be regarded as a function  $\varphi : P \longrightarrow V$ , such that  $\varphi(pg) = g^{-1}\varphi(p)$ ,  $\forall p \in P$ ,  $\forall g \in G$ . Let us denote the space of such functions on the total space of the principal bundle  $P$ , by  $C^\infty(P, V)_G$ .

A connection form  $A$  on the principle bundle  $(P, M, \pi)$  is a differential 1-form, with values in  $\mathfrak{g}$  – the Lie algebra of the Lie group  $G$ , such that for any  $g \in G$ :  $R_g^*(A) = \text{Ad}(g^{-1})(A)$ ; and for any  $u \in \mathfrak{g}$ :  $A(\tilde{u}) = u$ ; where  $\tilde{u}$  denotes the tangent vector

$$\tilde{u} \in T_p(P), \quad \tilde{u} = \frac{d}{dt}(p \cdot \exp(t \cdot u))$$

The connection form  $A$  defines a distribution of horizontal subspaces on the total space  $P$ :

$$H_p = \ker(A_p) \subset T_p(P), \quad \forall p \in P$$

This distribution is carried to the total space of the associated vector bundle, by the quotient mapping:  $P \times V \longrightarrow P_V = (P \times V)/G$ .

A vector field  $X$  on the manifold  $M$  can be lifted to a “horizontal” vector field  $\tilde{X}$  on the total space  $P$ , which is uniquely defined by the following conditions

$$\tilde{X}_p \in \ker(A_p), \quad \forall p \in P \text{ and } \pi'(\tilde{X}) = X$$

As it was mentioned, a section of the associated vector bundle  $s \in \Gamma(P_V)$  can be identified with an element  $\varphi_s \in C^\infty(P, V)_G$ . The covariant derivation of the section  $s$  is the section of the vector bundle  $(P_V, M, \pi_V)$ , corresponding to the function  $\varphi'_s(\tilde{X}) \in C^\infty(P, V)_G$ .

## 4 The Geometry of The Canonical Vector Bundle on Grassmann Manifold

Consider the set  $\Pi_m(\mathcal{H}) = \{(X, x) \mid X \in Gr_m(\mathcal{H}), x \in X\}$ , which is the subset of  $Gr_m(\mathcal{H}) \times \mathcal{H}$ . The set  $\Pi_m(\mathcal{H})$  together with the differential structure induced from  $Gr_m(\mathcal{H}) \times \mathcal{H}$  is a differential manifold. The mapping

$$p : \Pi_m(\mathcal{H}) \longrightarrow Gr_m(\mathcal{H}), \quad p(X, x) = X$$

is a vector bundle over the Grassmann manifold  $Gr_m(\mathcal{H})$ . The fiber at a point  $X \in Gr_m(\mathcal{H})$  is the vector space  $X \subset \mathcal{H}$ , itself. This vector bundle is known as the *canonical vector bundle* over the Grassmann manifold  $Gr_m(\mathcal{H})$ . For any point  $X \in Gr_m(\mathcal{H})$ , the local trivialization of this vector bundle, in the neighborhood of the point  $X$ , naturally arises from the coordinate system:

$$\Phi_X : Hom(X, X^\perp) \times X \longrightarrow \Pi_m(\mathcal{H}), \quad \Phi_X(\varphi, x) = (\Gamma(\varphi), x + \varphi(x))$$

If we regard the manifold  $Gr_m(\mathcal{H})$  as the manifold of orthonormal projectors,  $\mathcal{P}_m(\mathcal{H})$  (see 1), we can describe the total space  $\Pi_m(\mathcal{H})$ , as a submanifold of  $\mathcal{P}_m(\mathcal{H}) \times \mathcal{H}$  defined by the equation  $P(x) = x$ ,  $(P, x) \in \mathcal{P}_m(\mathcal{H}) \times \mathcal{H}$ . The latter implies that the tangent space of the manifold  $\Pi_m(\mathcal{H})$  is defined by the equation  $dP(x) + P(dx) = dx$ . More precisely: for any point  $Q = (P, x) \in \Pi_m(\mathcal{H})$ , where  $P \in \mathcal{P}_m(\mathcal{H}) \cong Gr_m(\mathcal{H})$  is a projector and  $x \in \text{Im}(P) = X$ ,

the tangent space  $T_Q(\Pi_m(\mathcal{H}))$  is the subspace of the space  $Hom(X, X^\perp) \times \mathcal{H}$  defined by the linear equation:  $\varphi(x) = (1 - P)u$ ,  $\varphi \in Hom(X, X^\perp)$ ,  $u \in \mathcal{H}$ . That is

$$T_{(X,x)}(\Pi_m(\mathcal{H})) = \{(\varphi, u) \in Hom(X, X^\perp) \times \mathcal{H} \mid \varphi(x) = \mathcal{P}(X^\perp)(u)\} \quad (12)$$

From this description of the tangent space of the total space of the canonical fiber bundle, easily follows that the vertical tangent subspace at a point  $(X, x) \in \Pi_m(\mathcal{H})$  is

$$Vert(X, x) = \{(0, u) \in Hom(X, X^\perp) \times \mathcal{H} \mid u \in X\}$$

One natural choice of a horizontal complement of the vertical subspace could be

$$Hor(X, x) = \{(\varphi, \varphi(x)) \mid \varphi \in Hom(X, X^\perp)\}$$

The projection operator on the vertical tangent subspace, along the horizontal tangent subspace is

$$P_{Vert} : T_{(X,x)}(\Pi_m(\mathcal{H})) \longrightarrow Vert(X, x), \quad P_{Vert}(\varphi, w) = (0, w - \varphi(x)) \quad (13)$$

**Proposition 1** *The distribution of horizontal subspaces*

$$Hor(X, x) \subset T_{(X,x)}(\Pi_m(\mathcal{H})), \quad (X, x) \in \Pi_m(\mathcal{H})$$

is a connection on the canonical vector bundle  $\Pi_m(\mathcal{H})$ . The corresponding covariant derivation acts on the space of sections of  $\Pi_m(\mathcal{H})$  as follows: any section  $s \in S(\Pi_m(\mathcal{H}))$  can be regarded as a function  $s : Gr_m(\mathcal{H}) \longrightarrow \mathcal{H}$ ; for any tangent vector  $\varphi \in T_X(Gr_m(\mathcal{H})) \cong Hom(X, X^\perp)$ , the covariant derivative of  $s$  by  $\varphi$  is  $\nabla_\varphi(s) = \mathcal{P}(X)(s'_X(\varphi)) = s'_X(\varphi) - \varphi(s(X))$ .

**Proof.** As it follows from the formula for the projection on the vertical tangent subspace (see 13), the action of the differential of the function  $s$  on the tangent vector  $\varphi$  and then its projection on the vertical tangent space is  $s'_X(\varphi) - \varphi(s(X))$ . As it follows from the definition of the tangent space of  $\Pi_m(\mathcal{H})$  (see 12), it is the same as the projection of  $s'_X(\varphi)$  on the subspace  $X$  along its orthogonal complement  $X^\perp$ .

It can be verified by direct calculations that the operation  $\nabla_\varphi(s) = S'(\varphi) - \varphi \circ s$  has the properties of covariant derivation. ■

Now consider the vector bundle  $q : \Pi_m^\perp(\mathcal{H}) \longrightarrow M$  the fiber of which at a

point  $X \in Gr_m(\mathcal{H})$  is  $X^\perp$  – the orthogonal complement of  $X$ . It is clear that the Whitney sum of two vector bundles  $\Pi_m(\mathcal{H}) \oplus \Pi_m^\perp(\mathcal{H})$  is the trivial vector bundle:  $Gr_m(\mathcal{H}) \times \mathcal{H}$ . As in the case of  $\Pi_m(\mathcal{H})$ , we can describe the total space  $\Pi_m^\perp(\mathcal{H})$  as a submanifold in  $\mathcal{P}_m(\mathcal{H}) \cong Gr_m(\mathcal{H}) \times \mathcal{H}$ :

$$\Pi_m^\perp(\mathcal{H}) = \{(P, y) \in \mathcal{P}_m(\mathcal{H}) \times \mathcal{H} \mid P(y) = 0\}$$

Differentiating the equation  $P(y) = 0$ , we obtain the equation for the tangent space of  $\Pi_m^\perp(\mathcal{H})$ :  $dP(y) + P(dy) = 0$ . This implies that the tangent space of  $\Pi_m^\perp(\mathcal{H})$  at a point  $(X, y)$ ,  $X \in Gr_m(\mathcal{H})$ ,  $y \in X^\perp$  is the subspace of  $Hom(X, X^\perp) \times \mathcal{H}$ :

$$T_{(X, y)}(\Pi_m^\perp(\mathcal{H})) = \{(\varphi, u) \in Hom(X, X^\perp) \times \mathcal{H} \mid \varphi^*(y) + \mathcal{P}(X)(u) = 0\} \quad (14)$$

From this description of the tangent space of  $\Pi_m^\perp(\mathcal{H})$ , follows that the vertical tangent subspace for the fiber bundle  $q : \Pi_m^\perp(\mathcal{H}) \longrightarrow M$  at a point  $(X, y)$  is

$$Vert(X, y) = \{(0, u) \in Hom(X, X^\perp) \times \mathcal{H} \mid u \in X^\perp\} \quad (15)$$

One choice of the complementar (“horizontal”) subspace is

$$Hor(X, y) = \{(\varphi, -\varphi^*(y)) \in Hom(X, X^\perp) \times \mathcal{H}\}$$

The discussion analogical to that in the proof of the proposition 1, shows that the covariant derivation corresponding to this distribution of horizontal subspaces is

$$\nabla_\varphi^\perp(s) = s'_X(\varphi) + \varphi^*(s(X)) = \mathcal{P}(X^\perp)(s'_X(\varphi))$$

for  $\varphi \in Hom(X, X^\perp) \cong T_X(Gr_m(\mathcal{H}))$  and  $s$  is a section of the vector bundle  $\Pi_m^\perp(\mathcal{H})$  (in this case regarded as a function  $s : Gr_m(\mathcal{H}) \longrightarrow \mathcal{H}$ ).

It is clear that  $\Pi_m(\mathcal{H}) \oplus \Pi_m^\perp(\mathcal{H}) = M \times \mathcal{H}$ . Using the definition of the covariant derivation on the sum of two vector bundles (see the formula 7), we obtain a covariant derivation on the trivial vector bundle  $M \times \mathcal{H}$ :  $\nabla^\mathcal{H} = \nabla \oplus \nabla^\perp$

$$\nabla_\varphi^\mathcal{H}(s) = s'_X(\varphi) - \varphi(s_1(X)) + \varphi^*(s_2(X)) \quad (16)$$

where  $X \in Gr_m(\mathcal{H})$ ,  $\varphi \in Hom(X, X^\perp) \cong T_X Gr_m(\mathcal{H})$ ,  $s_1$  is a section of  $\Pi_m(\mathcal{H})$ ,  $s_2$  is a section of  $\Pi_m^\perp(\mathcal{H})$  and  $s = s_1 + s_2$  is a section of  $Gr_m(\mathcal{H}) \times \mathcal{H}$  (that is, a function  $Gr_m(\mathcal{H}) \longrightarrow \mathcal{H}$ ).

The formula 16 for the covariant derivation on  $Gr_m(\mathcal{H}) \times \mathcal{H}$  implies that the corresponding connection form  $\mathcal{F}$  is a 1-form on the manifold  $Gr_m(\mathcal{H})$  with values in the Lie algebra of the group of unitary transformations of the Hilbert space  $\mathcal{H}$  (which is the Lie algebra antisymmetric operators on  $\mathcal{H}$ ):

$$\mathcal{F}_X(\varphi) = \begin{bmatrix} 0 & \varphi \\ -\varphi^* & 0 \end{bmatrix} \in \mathfrak{u}(\mathcal{H})$$

for  $X \in Gr_m(\mathcal{H})$  and  $\varphi \in T_X(Gr_m(\mathcal{H})) \cong Hom(X, X^\perp)$

As it was mentioned (see the formula 5), the projective representation of the Grassmann manifold gives rise of the 1-form with values in the space of symmetric operators

$$d\mathcal{P}(\varphi) = \begin{bmatrix} 0 & \varphi^* \\ \varphi & 0 \end{bmatrix}, \quad \varphi \in T_X(Gr_m(\mathcal{H})) \cong Hom(X, X^\perp)$$

Using this form, the connection form  $\mathcal{F}$ , can be written as a 1-form on  $\mathcal{P}_m(\mathcal{H}) \cong Gr_m(\mathcal{H})$ , as

$$\mathcal{F} = \mathcal{P}d\mathcal{P} + (\mathcal{P} - 1)d\mathcal{P} = 2\mathcal{P}d\mathcal{P} - d\mathcal{P} \quad (17)$$

Its differential (the curvature form) is  $d\mathcal{F} = 2d\mathcal{P}d\mathcal{P}$ , which is a 2-form with values in the Lie algebra of antisymmetric operators on the Hilbert space  $\mathcal{H}$

$$(d\mathcal{F})(\varphi, \psi) = 2 \begin{bmatrix} \varphi^*\psi - \psi^*\varphi & 0 \\ 0 & \varphi\psi^* - \psi\varphi^* \end{bmatrix}, \quad \varphi, \psi \in Hom(X, X^\perp)$$

Let us denote by  $\mathfrak{S}(\mathcal{H})$  the space of symmetric operators on the Hilbert space

$\mathcal{H}$ . The empedding  $\mathcal{P} : Gr_m(\mathcal{H}) \longrightarrow \mathfrak{S}(\mathcal{H})$  of the Grassmann manifold as the set of projectors, induces a mapping  $\mathcal{P}' : T(Gr_m(\mathcal{H})) \longrightarrow \mathfrak{S}(\mathcal{H}) \times \mathfrak{S}(\mathcal{H})$

$$\mathcal{P}'_X(\varphi) = (\mathcal{P}(X), \tilde{\varphi})$$

where

$$\tilde{\varphi} = \begin{bmatrix} 0 & \varphi^* \\ \varphi & 0 \end{bmatrix}$$

This mapping, according to the remark 3 induces a connection on the tangent bundle  $T(Gr_m(\mathcal{H}))$ , which itself, coincides with the connection  $\nabla^* \otimes \nabla^\perp$ , where  $\nabla$  is the connection on the canonical vector bundle  $\Pi_m(\mathcal{H})$  and  $\nabla^\perp$  is the connection on  $\Pi_m^\perp(\mathcal{H})$ .

## 5 Grassmann Manifold as an Orbit of the (co-)Adjoint Action and the Corresponding Symplectic Structure

Let  $L$  be a finite-dimensional Lie algebra over the field of complex or real numbers and  $L^*$  be its dual vector space. The linear mapping

$$B : L \wedge L \longrightarrow L, \quad B(x \wedge y) = [x, y]$$

induces the dual mapping between the dual spaces

$$B^* : L^* \longrightarrow L^* \wedge L^*, \quad B^*(\varphi)(x \wedge y) = \varphi([x, y])$$

The latter can be regarded as an antisymmetric covariant tensor field on the linear space  $L^*$ . Such a field defines a bracket on the algebra of smooth functions  $C^\infty(L^*)$ :

$$\{f, g\}(\varphi) = (df|_\varphi \wedge dg|_\varphi)(B^*(\varphi)) = \varphi([df|_\varphi, dg|_\varphi])$$

where the elements of the space  $L^{**}$ :  $df|_\varphi$  and  $dg|_\varphi$  are considered as elements of the vector space  $L$  which is canonically isomorphic to  $L^{**}$ . The algebra  $C^\infty(L^*)$  together with the above defined bracket is a Poisson algebra. Particularly, if we take any two elements  $x, y \in L$ , they can be regarded as linear functions on the vector space  $L^*$ :  $x(\varphi) = \varphi(x)$ ,  $\forall \varphi \in L^*$ ; and the Poisson bracket of these two linear functions is  $\{x, y\} = [x, y]$ .

If the vector space  $L$  is equipped with a scalar product  $\langle \cdot, \cdot \rangle$ , we can carry the Poisson structure from the space  $L^*$  to  $L$  by the linear mapping  $u \mapsto \langle u, \cdot \rangle : L \longrightarrow L^*$ . Under these conditions, an element  $x \in L$  can be considered as a linear function on the vector space  $L$ :  $\tilde{x}(u) = \langle x, u \rangle$ ,  $\forall u \in L$ . The Poisson bracket of such two functions is

$$\{\tilde{x}, \tilde{y}\}(u) = \langle [x, y], u \rangle = \widetilde{[x, y]}(u), \quad \forall u \in L$$



Now, assume that  $L$  is a Lie algebra of a Lie group  $G$  and the scalar product on the vector space  $L$  is invariant under the adjoint action of the group  $G$ :

$$\langle gug^{-1}, vvg^{-1} \rangle = \langle u, v \rangle, \quad \forall u, v \in L, \forall g \in G \quad (18)$$

Suppose that  $g = \exp(tw)$ ,  $w \in L, t \in \mathbb{R}$  is a one-parameter subgroup generated by an element  $w$ . Differentiating the equality 18 by the parameter  $t$  at the point  $t = 0$ , we obtain the following infinitesimal version of the equality 18

$$\langle [w, u], v \rangle + \langle u, [w, v] \rangle = 0 \quad (19)$$

For any function  $f \in C^\infty(L)$ , we denote by  $\text{ham}(f)$  the Hamiltonian vector field corresponding to the function  $f$  for the Poisson structure defined by  $\langle \cdot, \cdot \rangle$ . For any element  $x \in L$  and the corresponding linear function  $\tilde{x} = \langle x, \cdot \rangle$  on  $L$ , we have the following

$$\begin{aligned} d\tilde{y}(\text{ham}(\tilde{x})) &= \{\tilde{x}, \tilde{y}\} = \widetilde{[x, y]} \Rightarrow \\ \Rightarrow \forall u \in L : \tilde{y}(\text{ham}(\tilde{x}_u)) &= [x, y](u) = \langle [x, y], u \rangle \end{aligned}$$

Keeping in mind the equality 19, we obtain the following

$$\text{ham}(\tilde{x})_u = [x, u], \quad \forall u \in L$$

That is: for a Poisson structure defined by an adjoint-invariant metric on a Lie algebra  $L$ , the Hamiltonian vector field corresponding to a linear function  $\tilde{x} = \langle x, \cdot \rangle$ ,  $x \in L$  is  $\text{ham}(\tilde{x})_u = [x, u]$ ,  $\forall u \in L$ . Hence, we obtain that for any point  $u \in L$ , the subspace of the tangent space  $T(L) \cong L$ , generated by the Hamiltonian vector fields at the point  $u$  is

$$\text{Ham}_u = \{[x, u] \mid x \in L\}$$

The distribution of the subspaces of the tangent spaces:  $u \mapsto \text{Ham}_u$ ,  $u \in L$ , is integrable and the integral submanifolds of this distribution are the orbits of the adjoint action of the corresponding Lie group  $G$ . These orbits are exactly the symplectic leaves of the Poisson structure and as it should be, the restriction of the Poisson structure on each of them is nondegenerated. The corresponding symplectic form on any orbit of the adjoint action is

$$\omega(x; u, v) = \langle x, [u, v] \rangle, \quad x, u, v \in L \quad (20)$$

Any curve  $u : \mathbb{R} \longrightarrow L$  can be regarded as a time-dependent Hamiltonian:

$\tilde{u}_t(x) = \langle u(t), x \rangle$ ,  $t \in \mathbb{R}, x \in L$ ; and the corresponding Hamiltonian equation is  $\dot{x}(t) = [u(t), x(t)]$ . As the symplectic leaves of the Poisson structure are the orbits of the adjoint action, we have that any solution of the Hamiltonian equation lies in some orbit of the adjoint action.

Now consider the situation when the Lie algebra  $L$  is the Lie algebra of antisymmetric operators on the Hilbert space  $\mathcal{H}$ . This Lie algebra is the same as the Lie algebra of the Lie group  $U(\mathcal{H})$  – the group of unitary transformations of the Hilbert space  $\mathcal{H}$ . We denote this Lie algebra by  $\mathfrak{u}(\mathcal{H})$ . The mapping  $\mathcal{P}_m(\mathcal{H}) \longrightarrow \mathfrak{u}(\mathcal{H})$ ,  $P \mapsto -iP$  is an injection of the Grassmann manifold in the Lie algebra  $\mathfrak{u}(\mathcal{H})$ . The equality  $\mathcal{P}(g(X)) = g\mathcal{P}(X)g^{-1}$ ,  $\forall X \in Gr_m(\mathcal{H})$ ,  $\forall g \in U(\mathcal{H})$ , implies that the image of the above mapping –  $i\mathcal{P}_m(\mathcal{H})$ , is an orbit of the adjoint action of the Lie group  $U(\mathcal{H})$  on its Lie algebra –  $\mathfrak{u}(\mathcal{H})$ . Therefore, the Grassmann manifold can be regarded as an orbit of an adjoint action of a Lie group on its Lie algebra.

Consider the following scalar product on the vector space  $\mathfrak{u}(\mathcal{H})$ :

$$\langle u, v \rangle = \text{trace}(u^* \circ v), \quad u, v \in \mathfrak{u}(\mathcal{H})$$

For any unitary transformation  $g \in U(\mathcal{H})$ , we have

$$\langle gug^{-1}, gvg^{-1} \rangle = \text{trace}(gu^*vg^{-1}) = \text{trace}(u^*v) = \langle u, v \rangle, \quad \forall u, v \in \mathfrak{u}(\mathcal{H})$$

which implies that the scalar product on the space  $\mathfrak{u}(\mathcal{H})$  is invariant under the adjoint action of the group  $U(\mathcal{H})$ . Hence, we have the symplectic form on any orbit of the adjoint action, defined by the formula 20. Particularly, for the embedded Grassmann manifold in  $\mathfrak{u}(\mathcal{H})$ , we have

$$\omega(iP; \Phi, \Psi) = \text{trace}(iP \circ [\Phi, \Psi]) \quad (21)$$

To obtain more explicit expression for the differential 2-form  $\omega$ , at a point  $P \in \mathcal{P}_m(\mathcal{H}) \cong Gr_m(\mathcal{H})$ , consider the decomposition of the operators  $\Phi$  and  $\Psi$  corresponding to the decomposition  $\mathcal{H} = \text{Im}(P) \oplus \text{Im}(P)^\perp$ :

$$\Phi = \begin{bmatrix} 0 & i\varphi^* \\ i\varphi & 0 \end{bmatrix}, \quad \Psi = \begin{bmatrix} 0 & i\psi^* \\ i\psi & 0 \end{bmatrix}$$

where  $\varphi, \psi : \text{Im}(P) \longrightarrow \text{Im}(P)^\perp$ . From the formula 21 easily follows the following expression for the symplectic form:

$$\omega(X; \varphi, \psi) = i \cdot \text{trace}(\varphi^* \psi + \psi^* \varphi) \quad (22)$$

for  $X \in Gr_m(\mathcal{H})$  and  $\varphi, \psi \in T_X(Gr_m(\mathcal{H})) \cong Hom(X, X^\perp)$

which can be written as  $w = i \text{trace}(P \cdot dP \wedge dP)$ .

As it was mentioned for the case of a general Lie algebra, any element  $u \in \mathfrak{u}(\mathcal{H})$  can be considered as a function  $\tilde{u} : Gr_m(\mathcal{H}) \longrightarrow \mathbb{R}$

$$\tilde{u}(P) = i \cdot \text{trace}(u^* \circ P) = -i \cdot \text{trace}(u \circ P), \quad P \in \mathcal{P}_m(\mathcal{H}) \cong Gr_m(\mathcal{H})$$

and the corresponding Hamiltonian vector field on the manifold  $Gr_m(\mathcal{H})$  is

$$\text{ham}(\tilde{u})_P = [u, P], \quad P \in \mathcal{P}_m(\mathcal{H}) \cong Gr_m(\mathcal{H}) \quad (23)$$

This vector field is the same as that one generated by the action of the group  $U(\mathcal{H})$ .

## 6 The Canonical Principal Bundle on Grassmann Manifold and the Canonical Connection

Consider the set of all orthonormal  $m$ -frames in the Hilbert space  $\mathcal{H}$ . This set is known as the Stiefel manifold (see [8]) for the complex Hilbert space  $\mathcal{H}$ . It is clear that the Stiefel manifold of orthonormal  $m$ -frames can be identified with the set of all isometric mappings from the complex vector space  $\mathbb{C}^m$  to the Hilbert space  $\mathcal{H}$ . We denote this set of all isometric mappings from  $\mathbb{C}^m$  to  $\mathcal{H}$  by  $Isom(\mathbb{C}^m, \mathcal{H})$ . This set equipped with the topology and the differential structure of the subset of the vector space  $Hom(\mathbb{C}^m, \mathcal{H})$ , is a manifold, and can be described as follows

$$Isom(\mathbb{C}^m, \mathcal{H}) = \{\varphi \in Hom(\mathbb{C}^m, \mathcal{H}) \mid \varphi^* \varphi = 1\}$$

There is a natural mapping from the manifold  $Isom(\mathbb{C}^m, \mathcal{H})$  to the manifold  $Gr_m(\mathcal{H})$  defined as

$$\pi : Isom(\mathbb{C}^m, \mathcal{H}) \longrightarrow Gr_m(\mathcal{H}), \quad \pi(\varphi) = \text{Im}(\varphi)$$

Recall that for any  $X \in Gr_m(\mathcal{H})$ , the symbol  $\mathcal{P}(X)$  denotes the operator on the Hilbert space  $\mathcal{H}$  that is the orthogonal projection on the subspace  $X$ .

**Lemma 1** For any  $\varphi \in Isom(\mathbb{C}^m, \mathcal{H})$ , we have that  $\mathcal{P}(Im(\varphi)) = \varphi \circ \varphi^*$  (see [4], [5], [6])

**Proof.** The proof consists of the following two steps: firstly we check that for any  $x \in Im(\varphi)$ :  $(\varphi\varphi^*)(x) = x$ , and then we check that for any  $y \in Im(\varphi)^\perp$ :  $(\varphi\varphi^*)(y) = 0$ .

For  $x \in Im(\varphi)$ , we have the following:

$$x = \varphi(u) \Rightarrow (\varphi\varphi^*)(x) = (\varphi \underbrace{\varphi^*\varphi}_1)(u) = \varphi(u) = x$$

For  $y \in Im(\varphi)^\perp$ , we have:

$$\begin{aligned} \forall u \in \mathcal{H} : \langle (\varphi\varphi^*)(y), u \rangle &= \langle \varphi^*(y), \varphi^*(u) \rangle = \langle y, \underbrace{(\varphi\varphi^*)(y)}_{\in Im(\varphi)} \rangle = 0 \Rightarrow \\ &\Rightarrow (\varphi\varphi^*)(y) = 0 \end{aligned}$$

■

This implies that that, as the manifold  $Gr_m(\mathcal{H})$  is identified with the space of projectors  $\mathcal{P}_m(\mathcal{H})$ , in some cases, we can substitute the mapping

$$\pi : Isom(\mathbb{C}^m, \mathcal{H}) \longrightarrow Gr_m(\mathcal{H})$$

with the mapping

$$\pi : Isom(\mathbb{C}^m, \mathcal{H}) \longrightarrow \mathcal{P}_m(\mathcal{H}), \quad \varphi \mapsto \varphi\varphi^*$$

The latter we also denote by  $\pi$ .

For any  $X \in Gr_m(\mathcal{H})$ , consider any isometric map  $\varphi : \mathbb{C}^m \longrightarrow X$ . It is clear that  $\pi(\varphi) = X$ . Hence the mapping  $\pi$  is surjective and for each  $X \in Gr_m(\mathcal{H})$  we have that

$$\pi^{-1}(X) = \{\varphi \in Isom(\mathbb{C}^m, \mathcal{H}) \mid Im(\varphi) = X\}$$

**Lemma 2** Let  $\{u_1, \dots, u_m\}$  be any basis of a Hermitian vector space  $X$ . There exists one and only one orthogonal basis  $\{e_1, \dots, e_m\}$  in the space  $X$ , such that for any  $i = 1, \dots, m$ :  $Span \langle e_1, \dots, e_i \rangle = Span \langle u_1, \dots, u_i \rangle$  and  $e_1 \wedge \dots \wedge e_i = k \cdot u_1 \wedge \dots \wedge u_i$ , where  $k > 0$  (i.e., the orientations of the frames  $\{e_1, \dots, e_i\}$  and  $\{u_1, \dots, u_i\}$  coincide).

**Proof.** We prove this lemma by induction. For  $m = 1$ , it is clear that the basis  $e_1$  should be  $u_1/\|u_1\|$ . If the statement is true for  $m - 1$  then consider a vector of the type  $e' = x_1e_1 + \dots + x_{m-1}e_{m-1} + u_m$ . The system of equations  $\langle e', e_i \rangle = 0$ ,  $i = 1, \dots, m - 1$  gives the values of the numbers  $x_1, \dots, x_{m-1}$ . After this, we can obtain the vector  $e_m$  by normalizing the vector  $e'$ :  $e_m = c \cdot e'$ ; so that  $\|e_m\| = 1$  and the orientation of the frame  $\{e_1, \dots, e_m\}$  be the same as the orientation of  $\{u_1, \dots, u_m\}$  and it is clear that this could be done in exactly one way. ■

The statement of the above lemma implies that for any isomorphism  $f : \mathbb{C}^m \longrightarrow X$ , there is exactly one linear transformation  $I(f) : X \longrightarrow X$ , such that  $\Phi(f) = I(f) \circ f : \mathbb{C}^m \longrightarrow X$  is an isometric mapping and  $\text{Span} \langle \Phi(f)(e_1), \dots, \Phi(f)(e_i) \rangle = \text{Span} \langle f(e_1), \dots, f(e_i) \rangle$  and the orientations of the frames  $\{\Phi(f)(e_1), \dots, \Phi(f)(e_i)\}$  and  $\{f(e_1), \dots, f(e_i)\}$  coincide, for  $i = 1, \dots, m$ , where  $e_1, \dots, e_m$  is the natural basis in the complex vector space  $\mathbb{C}^m$ . We call the mapping  $\Phi(f)$  the *isometrization* of the isomorphism  $f$ .

By using of the above construction we can define a local trivialization of  $\pi : \text{Isom}(\mathbb{C}^m, \mathcal{H}) \longrightarrow \text{Gr}_m(\mathcal{H})$  as follows: for any point  $X \in \text{Gr}_m(\mathcal{H})$ , consider the mapping

$$\Phi_X : \text{Isom}(\mathbb{C}^m, X) \times \text{Hom}(X, X^\perp) \longrightarrow \text{Isom}(\mathbb{C}^m, \mathcal{H})$$

$$\Phi_X(\varphi, f) = \Phi((1 + f)\varphi)$$

It is clear that the mapping  $(1 + f)\varphi$  is a monomorphism, the image of which is the subspace  $\Gamma(f) \in \text{Gr}_m(\mathcal{H})$ , and  $\Phi((1 + f)\varphi)$  is the isometrization of the mapping  $(1 + f)\varphi$ .

Hence, we obtain that  $\pi : \text{Isom}(\mathbb{C}^m, \mathcal{H}) \longrightarrow \text{Gr}_m(\mathcal{H})$  is a locally trivial fiber bundle.

Consider the following right action of  $U(m)$  – the group of unitary transformations of  $\mathbb{C}^m$ , on the space  $\text{Isom}(\mathbb{C}^m, \mathcal{H})$ :

$$\forall g \in U(m) \text{ and } \forall \varphi \in \text{Isom}(\mathbb{C}^m, \mathcal{H}) \text{ let } \varphi \mapsto \varphi \circ g$$

It is clear that  $\pi(\varphi g) = \pi(\varphi)$ .

If  $\varphi$  and  $\psi$  are in one and the same fiber  $\pi^{-1}(X)$  for  $X \in \text{Gr}_m(\mathcal{H})$ , then  $\varphi\varphi^* = \psi\psi^*$ , which implies that  $\varphi = \psi \underbrace{\psi^*\varphi}_g = \psi g$ . For  $g = \psi^*\varphi$  we have

$g^*g = \varphi^* \underbrace{\psi\psi^*}_1 \varphi = \varphi^*\varphi = 1$ , which implies that  $g$  is the element of the unitary group  $U(m)$ . We obtain that the right action of the group  $U(m)$  on the fiber of the bundle  $\pi : Isom(\mathbb{C}^m, \mathcal{H}) \longrightarrow Gr_m(\mathcal{H})$  is transitive and effective. In other words, this fiber bundle is a principal bundle with the structure group  $U(m)$ . The associated vector bundle  $(Isom(\mathbb{C}^m, \mathcal{H}) \times \mathbb{C}^m)/U(m)$  where

$$(\varphi, x) \sim (\varphi \circ g, g^{-1}(x)), \quad \forall (\varphi, x) \in Isom(\mathbb{C}^m, \mathcal{H}) \times \mathbb{C}^m \text{ and } \forall g \in U(m)$$

is isomorphic to the canonical vector bundle  $\Pi_m(\mathcal{H})$  via the mapping

$$[\varphi, x] \mapsto \varphi(x)$$

The subset  $Isom(\mathbb{C}^m, \mathcal{H})$  in the space  $Hom(\mathbb{C}^m, \mathcal{H})$  is defined by the equation  $\varphi^*\varphi = 1$ , which implies that the tangent space of  $Isom(\mathbb{C}^m, \mathcal{H})$  can be described by the equation

$$d\varphi^*\varphi + \varphi^*d\varphi = 0 \quad (24)$$

Otherwise

$$T_\varphi(Isom(\mathbb{C}^m, \mathcal{H})) = \{u \in Hom(\mathbb{C}^m, \mathcal{H}) \mid u^*\varphi + \varphi^*u = 0\} \quad (25)$$

Differentiating the mapping  $\pi(\varphi) = \varphi\varphi^*$  from  $Isom(\mathbb{C}^m, \mathcal{H})$  to  $Gr_m(\mathcal{H})$ , we obtain

$$d\pi = d\varphi\varphi^* + \varphi d\varphi^* \quad (26)$$

**Proposition 2** *The vertical tangent subspace of the fiber bundle*

$$\pi : Isom(\mathbb{C}^m, \mathcal{H}) \longrightarrow Gr_m(\mathcal{H})$$

*at a point  $\varphi \in Isom(\mathbb{C}^m, \mathcal{H})$  is the subspace of the tangent space  $T_\varphi$  consisting of such  $v : \mathbb{C}^m \longrightarrow \mathcal{H}$  that  $Im(v) \subset Im(\varphi)$ .*

**Proof.** By definition, the vertical tangent space at a point  $\varphi \in Isom(\mathbb{C}^m, \mathcal{H})$  is the kernel of the mapping  $\pi'(\varphi)$ . Hence, it is described by the following system of linear equations

$$\begin{cases} v^*\varphi + \varphi^*v = 0 & - \text{Tangent} \\ v\varphi^* + \varphi v^* = 0 & - \text{Vertical} \end{cases}$$

The multiplication of the second one by  $\varphi$  from the right, gives  $v + \varphi v^* \varphi = 0$ . Substitute the term  $v^* \varphi$  by  $-\varphi^* v$ , from the first equation, gives the following equation  $v - \varphi \varphi^* v = 0$ , or equivalently  $(1 - \mathcal{P}(\text{Im}(\varphi)))v = 0$ . But it is clear that the operator  $(1 - \mathcal{P}(\text{Im}(\varphi)))$  is the orthogonal projection operator on the subspace  $\text{Im}(\varphi)^\perp$ . This implies that  $\text{Im}(v) \subset \text{Im}(\varphi)$ . ■

For a given subspace  $X \subset \mathcal{H}$ , any linear mapping  $w : \mathbb{C}^m \longrightarrow \mathcal{H}$  is uniquely decomposed as a direct sum:  $w = v + u$ , with  $\mathbb{C}^m \xrightarrow{v} X$  and  $\mathbb{C}^m \xrightarrow{u} X^\perp$ . The above description of the vertical tangent space for the fiber at a point  $X \in \text{Gr}_m(\mathcal{H})$  suggests the following natural choice of its complementary (horizontal) subspace

$$\text{Hor}_X = \{u \in \text{Hom}(\mathbb{C}^m, \mathcal{H}) \mid \text{Im}(u) \subset X^\perp\}$$

For any  $\varphi \in \text{Isom}(\mathbb{C}^m, \mathcal{H})$  with  $\text{Im}(\varphi) = X$ , we have that  $\ker(\varphi) = X^\perp$ , therefore:  $\varphi^* u = 0$ , which implies that  $(\varphi^* u)^* = 0$  and finally:  $u^* \varphi + \varphi^* u = 0$ . Therefore, the space  $\text{Hor}_X$  automatically is subspace of the tangent space  $T_\varphi(\text{Isom}(\mathbb{C}^m, \mathcal{H}))$ .

The distribution of the horizontal subspaces  $\varphi \mapsto \text{Hor}_\varphi$  can be described by the equation  $\varphi^* d\varphi = 0$ . Consider the following differential form  $\mathcal{A} = \varphi^* d\varphi$  on the manifold  $\text{Isom}(\mathbb{C}^m, \mathcal{H})$ .

**Proposition 3** *The differential form  $\mathcal{A} = \varphi^* d\varphi$  takes its values in the Lie algebra of the Lie group  $U(m)$ , and is a connection form on the total space of the  $U(m)$ -principal bundle  $\pi : \text{Isom}(\mathbb{C}^m, \mathcal{H}) \longrightarrow \text{Gr}_m(\mathcal{H})$ .*

**Proof.** From the equation for the tangent space of  $\text{Isom}(\mathbb{C}^m, \mathcal{H})$ :  $\varphi^* d\varphi + d\varphi^* \varphi = 0$ , follows that the differential form  $\mathcal{A}$  takes its values in the Lie algebra of antisymmetric matrices:  $\mathcal{A}^* = d\varphi^* \varphi = -\varphi^* d\varphi = -\mathcal{A}$ .

The next step is to verify the transformation rule for the differential form  $\mathcal{A}$  under the right action of the unitary group  $U(m)$ . For any  $g \in U(m)$ , we have the following

$$R_g(\varphi) = \varphi g \Rightarrow R_g^*(\mathcal{A}) = g^* \varphi^* d(\varphi g) = g^{-1} \varphi^* d\varphi g = g^{-1} \mathcal{A} g = \text{Ad}(g^{-1})(\mathcal{A})$$

which shows that the transformation rule for the differential form  $\mathcal{A}$  is in accordance with the requirement for the connection forms.

For any fixed point  $\varphi \in \text{Isom}(\mathbb{C}^m, \mathcal{H})$ , consider the mapping

$$m_\varphi : U(m) \longrightarrow \text{Isom}(\mathbb{C}^m, \mathcal{H}), \quad m_\varphi(g) = \varphi g, \quad \forall g \in U(m)$$

Its differential at the point  $g = 1$  is the linear mapping from the Lie algebra  $\mathfrak{u}(m)$  to the tangent space  $T_\varphi(Isom(\mathbb{C}^m, \mathcal{H}))$ :  $m'_\varphi(1)(u) = \varphi u$ ,  $\forall u \in \mathfrak{u}(m)$ . Consider the value of the differential form  $\mathcal{A}$  on the vertical tangent vector  $m'_\varphi(1)(u) = \varphi u$ . We obtain  $\mathcal{A}(\varphi u) = \underbrace{\varphi^* \varphi}_1 u = u$ . The latter equality is also in accordance with the requirement for the connection forms. ■

Let  $\xi \in T_\varphi(Isom(\mathbb{C}^m, \mathcal{H}))$  be a horizontal tangent vector. That is:  $\xi \in Hom(\mathbb{C}^m, \mathcal{H})$  and  $Im(\xi) \subset Im(\varphi)^\perp$ . Its image in  $T_{\pi(\varphi)}Gr_m(\mathcal{H})$  by the differential of the projection map  $\pi$  is

$$\pi'(\varphi)(\xi) = \xi\varphi^* + \varphi\xi^* = \begin{bmatrix} 0 & \varphi\xi^* \\ \xi\varphi^* & 0 \end{bmatrix} : Im(\varphi) \oplus Im(\varphi)^\perp \longrightarrow Im(\varphi) \oplus Im(\varphi)^\perp$$

Otherwise, we can consider  $\xi$  as a linear mapping from  $\mathbb{C}^m$  to  $Im(\varphi)^\perp$ ,  $\varphi$  as a linear mapping from  $\mathbb{C}^m$  to  $Im(\varphi)$ , and  $\pi'(\varphi)(\xi)$ , as the composition

$$\xi \circ \varphi^* : Im(\varphi) \xrightarrow{\varphi^*} \mathbb{C}^m \xrightarrow{\xi} Im(\varphi)^\perp$$

which is the element of  $Hom(Im(\varphi), Im(\varphi)^\perp) \cong T_{Im(\varphi)}(Gr_m(\mathcal{H}))$ . Vice versa, any tangent vector  $\mu \in Hom(X, X^\perp)$  at a point  $X \in Gr_m(\mathcal{H})$ , could be lifted to the corresponding horizontal space at a point  $\varphi \in Isom(\mathbb{C}^m, \mathcal{H})$  as  $\widetilde{\mu}u = \mu\varphi$ , which is a mapping from  $\mathbb{C}^m$  to  $X^\perp$  (i.e., the element of the horizontal subspace in  $T_\varphi(Isom(\mathbb{C}^m, \mathcal{H}))$ ). It is clear that  $\widetilde{\xi\varphi^*} = \xi \underbrace{\varphi^* \varphi}_1 = \xi$ .

## 7 Geometric Control Theory and the Holonomy Algebra of the Connection Form $\varphi^*d\varphi$

Let us recall the following well known result from the theory of connections and their holonomies on principal bundles.

**Theorem 1 (Ambrous, Singer).** *Let  $A$  be a connection form on a principle bundle  $\pi : P \longrightarrow M$ . For any point  $p \in P$ , the holonomy algebra  $\Phi_p$  at the point  $p$  is the algebra spanned on the set of elements of the type  $\Omega(X, Y)$ , where  $\Omega$  is the curvature form for the connection form  $A$  and  $X$  and  $Y$  are horizontal tangent vectors of  $P$ , at such points  $q$  that can be connected with the point  $p$  by a horizontal curve.*



We use the statement of this theorem to investigate the holonomy algebra of the connection form  $\mathcal{A} = \varphi^* d\varphi$  on the canonical principal bundle over the Grassmann manifold.

**Proposition 4** *For any point  $\varphi \in Isom(\mathbb{C}^m, \mathcal{H})$ , the holonomy algebra of the connection  $\mathcal{A} = \varphi^* d\varphi$  at this point coincides with the Lie algebra of the Lie group  $U(\mathcal{H})$ .*

**Proof.** As it was shown, the horizontal subspace corresponding to the connection form  $\mathcal{A}$ , at a point  $\varphi \in Isom(\mathbb{C}^m, \mathcal{H})$  is

$$Hor(\varphi) = \{u \in Hom(\mathbb{C}^m, \mathcal{H}) \mid \text{Im}(u) \subset \text{Im}(\varphi)^\perp\}$$

and the value of the curvature form on a pair of horizontal vectors  $(u, v)$  at the point  $\varphi$  is

$$\Omega(u, v) = (d\mathcal{A})(u, v) = \frac{1}{2}(u^*v - v^*u)$$

We show that for any antisymmetric operator  $w : \mathbb{C}^m \longrightarrow \mathbb{C}^m$ , can be found such a pair of linear maps  $u, v : \mathbb{C}^m \longrightarrow \text{Im}(\varphi)^\perp$  that  $w = \frac{1}{2}(u^*v - v^*u)$ . In this case we assume that the Hilbert space is infinite-dimensional (or, at least has a dimension as big as we need). So, we can choose an  $m$ -dimensional complex subspace in  $\text{Im}(\varphi)^\perp$ . Fix any basis in  $X$ , after which it is identified with  $\mathbb{C}^m$ . If we take  $u = 1 : \mathbb{C}^m \longrightarrow X \cong \mathbb{C}^m$ , then from the equation  $w = \frac{1}{2}(v - v^*)$ , easily follows the solution  $v = w$ . ■

Hence, for the Grassmann manifold of an infinite-dimensional Hilbert space, it is not necessary to use the full strength of the theorem 7, because the  $\text{Im}(\Omega)$ , only in one point of the total space  $Isom(\mathbb{C}^m, \mathcal{H})$ , covers all the Lie algebra  $\mathfrak{u}(m)$ , even without spanning. The same is true for a finite-dimensional case, but when  $\dim(\text{Im}(\varphi)^\perp) \geq m$ . In the case when the Hilbert space is finite-dimensional and  $\dim(\text{Im}(\varphi)^\perp) < m$ , things are not so simple.

**Proposition 5** *For any integer  $n > m$  and any  $\psi \in Isom(\mathbb{C}^m, \mathbb{C}^n)$ , the holonomy algebra of the connection form  $\mathcal{A} = \varphi^* d\varphi$ , at the point  $\psi$ , coincides with the entire Lie algebra  $\mathfrak{u}(m)$ .*

**Proof.** Let us assume, that the antisymmetric operator  $w : \mathbb{C}^m \longrightarrow \mathbb{C}^m$  is diagonal

$$w = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & a_m \end{pmatrix}, \quad \text{where } a_i \in \mathbf{i} \cdot \mathbb{R}, i = 1, \dots, m$$

It is sufficient to consider the case when  $n = m + 1$ . We have that  $w = \sum_{i=1}^m w_i$ , where

$$w_i = \begin{pmatrix} 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & a_i & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & \cdot & \cdot & 0 \end{pmatrix}, \quad i = 1, \dots, m$$

For each  $w_i$  consider the pair of linear mappings  $u_i, v_i : \mathbb{C}^m \longrightarrow \mathbb{C}$ , where  $u_i = (\underbrace{0, \dots, 0}_{i-1}, 1, 0, \dots, 0)$  and  $v_i = (\underbrace{0, \dots, 0}_{i-1}, a_i, 0, \dots, 0)$ . For these mappings we have that  $w_i = \frac{1}{2}(u_i^* v_i - v_i^* u_i)$ . Hence, we obtain that any diagonal matrix  $w$  can be represented as  $w = \sum_i \Omega(u_i, v_i)$ , where  $u_i, v_i : \mathbb{C}^m \longrightarrow \mathbb{C}$ . Now, recall that for any antisymmetric operator  $w : \mathbb{C}^m \longrightarrow \mathbb{C}^m$  there exists a unitary transformation  $\tau : \mathbb{C}^m \longrightarrow \mathbb{C}^m$ , such that  $\tau w \tau^{-1}$  is diagonal. If  $\tau w \tau^{-1} = \sum_i \Omega(u_i, v_i)$ , then consider the operators  $\tilde{u}_i = u_i \tau$  and  $\tilde{v}_i = v_i \tau$ . We have that  $w = \sum_i \Omega(\tilde{u}_i, \tilde{v}_i)$ . ■

Consider the holonomy of the connection  $\mathcal{A} = \varphi^* d\varphi$  from the point of view of the geometric control theory. First of all, let us recall some definitions and facts from the geometric control theory (see [7]).

**Definition 2 (Control Group).** *Let  $I$  be any non-empty set. Consider the set of finite sequences of the type  $((t_1, i_1)(t_2, i_2), \dots, (t_p, i_p))$  with entries from the set  $\mathbb{R} \times I$ . Introduce the following reduction rules*

- *In any such sequence, the entries of the type  $(0, k)$  are removed;*
- *If  $i_k = i_{k+1}$ , then the segment  $(t_k, i_k)(t_{k+1}, i_{k+1})$  is replaced by the item  $(t_k + t_{k+1}, i_k)$ .*

*Applying the above reduction procedures, to any sequence, we obtain a non-reducible sequence, after a finite number of steps. The set of non-reducible sequences of the elements of the set  $\mathbb{R} \times I$  is called the **control set** corresponding to the set  $I$ . We denote the set of such sequences by  $C(I)$ . The elements of the set  $C(I)$  are called the **controls**.*

From any two elements of the control set  $C(I)$ ,  $s_1$  and  $s_2$ , we can construct a new one by concatenation  $s_1 s_2$  and then by reduction of the result –  $Red(s_1 s_2)$ . The mapping from the set  $C(I) \times C(I)$  to  $C(I)$ :

$$(s_1, s_2) \mapsto Red(s_1 s_2)$$

defines a group structure on the control set  $C(I)$ . The uital element in this group is the **empty** sequence. Further, the product of the elements  $s_1$  and  $s_2$  we denote by  $s_1 s_2$  or  $s_1 \cdot s_2$ .

For any control  $s = ((t_1, i_1), \dots, (t_n, i_n)) \in C(I)$  and a real number  $a \in \mathbb{R}$ , define the control  $a \cdot s$  as

$$a \cdot s = ((at_1, i_1), \dots, (at_n, i_n))$$

Any set of controls  $\{s_1, s_2, \dots, s_p\} \subset C(I)$ , defines a mapping

$$\phi_{s_1 s_2 \dots s_p} : \mathbb{R}^p \longrightarrow C(I)$$

as

$$(a_1, \dots, a_p) \mapsto (a_1 \cdot s_1, \dots, a_p \cdot s_p), \quad \forall (a_1, \dots, a_p) \in \mathbb{R}^p$$

Define the topology on the set  $C(I)$  as the strongest topology for which, all the mappings  $\{\phi_{s_1 \dots s_p} \mid p \in \mathbb{N}, s_i \in C(I), i = 1, \dots, p\}$  are continuous.

The differential structure on the topological space  $C(I)$  is defined as follows: any map  $f : C(I) \longrightarrow \mathbb{R}$  is differentiable if and only if all the maps  $f \circ \phi_{s_1 \dots s_p} : \mathbb{R}^p \longrightarrow \mathbb{R}$ , are differentiable.

**Definition 3** A **dynamical polysystem** on a smooth manifold  $M$ , controlled by the group of control  $C(I)$  is called a smooth left action of the group  $C(I)$  on the manifold  $M$ . For any point  $x \in M$ , the set  $C(I)x = \{sx \mid s \in C(I)\}$  is called the orbit of the point  $x$ .

Any element  $i \in I$  defines a one parameter group of diffeomorphisms

$$\varphi_t^i : M \longrightarrow M, \quad \varphi_t^i(x) = (t, i)x$$

which, itself, generates a vector field  $X^i$  on the manifold  $M$ . The family of vector fields  $\{X_i \mid i \in I\}$  is called the *infinitesimal transformations* of the dynamical polysystem  $(M, C(I))$ .

Conversely, for any family of vector fields  $\{X_i \mid i \in I\}$  on a smooth manifold  $M$ , the formula

$$(t_1, i_1) \cdots (t_n, i_n)x = (\varphi_{t_1}^{i_1} \circ \cdots \circ \varphi_{t_n}^{i_n})(x)$$

where  $\varphi_{t_1}^{i_1}, \dots, \varphi_{t_n}^{i_n}$  are the elements of the one-parameter groups of diffeomorphisms corresponding to the vector fields  $X_{i_1}, \dots, X_{i_n}$ , defines a dynamical polysystem controlled by the group  $C(I)$ .

To summarize, we can say that there is a one-to-one correspondence between the dynamical polysystems and the families of vector fields.

Further, for any subset of vector fields  $X$  on a smooth manifold  $M$ , the dynamical polysystem controlled by the group  $C(X)$  will be denoted also by  $C(X)$ .

One of the purposes of the geometric control theory is the investigation of the accessibility problem: for a given dynamical polysystem  $C(X)$ , corresponding to some family of vector fields  $X$ , find the orbit of a point  $m \in M$  —  $C(X)m$ . Let us formulate the following two fundamental theorems about the structure of the orbit of a dynamical polysystem, corresponding to a family of vector fields.

**Theorem 2 (Orbits Theorem. Nagano-Sussmann).** *Let  $X$  be a family of vector fields on a smooth manifold  $M$  and  $m$  is a point on  $M$ . Then:*

1. *The orbit  $C(X)m$  is an immersed submanifold in the manifold  $M$ ;*
2. *The tangent space  $T_x(C(X)m)$  at a point  $x \in C(X)m$  is the vector space generated by the set of vectors  $\{Ad(s)(u_x) \mid s \in C(X), u \in X\}$ .*

For any family of vector fields  $X$ , let us denote by  $Lie(X)$  the minimal submodule of  $C^\infty(M)$ -module of vector fields on the manifold  $M$ , containing the family  $X$  and closed under the operation of Lie bracket.

**Definition 4** *A family of vector fields  $X$  is called **completely nonholonomic** or **bracket-generating** if for each point  $m \in M$  we have that  $Lie(X)_m = T_m(M)$ .*

The next fundamental theorem of the geometric control theory is, practically, a corollary of the previous one.

**Theorem 3 (Rashevsky-Chow).** *Let  $M$  be a connected smooth manifold, and let  $X$  be a family of vector fields on the manifold  $M$ . If the family  $X$  is completely nonholonomic (i.e.,  $Lie(X)_m = T_m(M)$ ,  $\forall m \in M$ ) then the orbit  $C(X)m$  coincides with the entire manifold  $M$ , for each point  $m \in M$ .*

Let  $V$  be a complex subspace of the Hilbert space  $\mathcal{H}$ . As before,  $U(\mathcal{H})$  is the group of unitary transformations of the Hilbert space  $\mathcal{H}$  and  $\mathfrak{u}(V)$  is the Lie algebra of the Lie group of unitary transformations of the subspace  $V$ . Consider the following  $\mathfrak{u}(V)$ -valued 1-form on the Lie group  $U(\mathcal{H})$ :

$$\mathcal{B} = \iota^*(g^*dg)\iota$$

where  $\iota$  denotes the natural immersion mapping  $\iota : V \hookrightarrow \mathcal{H}$  and  $\iota^*$  is its dual  $\iota^* : \mathcal{H} \longrightarrow V$ . The value of the form  $\mathcal{B}$  can be interpreted as the  $(g^{-1}dg)_{VV}$  component of the operator  $g^{-1}dg \in \mathfrak{u}(\mathcal{H})$  in the decomposition

$$g^{-1}dg = \begin{bmatrix} (g^{-1}dg)_{VV} & (g^{-1}dg)_{V^\perp V} \\ (g^{-1}dg)_{VV^\perp} & (g^{-1}dg)_{V^\perp V^\perp} \end{bmatrix}$$

corresponding to the decomposition of the Hilbert space  $\mathcal{H} = V \oplus V^\perp$ . The differential form  $\mathcal{B}$  is left-invariant: for any  $a \in U(\mathcal{H})$  we have the following

$$L_a^*(\mathcal{B}) = \iota^*(g^*a^*adg)\iota = \iota^*(g^*dg)\iota = \mathcal{B}$$

Consider the distribution of tangent subspaces on the manifold  $U(\mathcal{H})$  defined by the equation  $\mathcal{B} = 0$ , i.e., the subspace of the tangent space at a point  $u \in U(\mathcal{H})$  is the kernel of the 1-form  $\mathcal{B}$  at the point  $u$ . As the differential form  $\mathcal{B}$  is left-invariant, the distribution  $X$  is left-invariant too:  $X_{gu} = L'_g(X_u)$ . Therefore, the distribution  $X$  is of a constant rank. But  $X$  is not an involutive distribution. It follows from the definition of the form  $\mathcal{B}$ : the subset of the tangent space  $X_1 \in T_1(U(\mathcal{H}))$ , consists of the operators of the form

$$f = \begin{bmatrix} 0 & -\varphi^* \\ \varphi & 0 \end{bmatrix} : \mathcal{H} = V \oplus V^\perp \longrightarrow \mathcal{H} = V \oplus V^\perp$$

If we have two such operators

$$f_1 = \begin{bmatrix} 0 & -\varphi_1^* \\ \varphi_1 & 0 \end{bmatrix} \text{ and } f_2 = \begin{bmatrix} 0 & -\varphi_2^* \\ \varphi_2 & 0 \end{bmatrix}$$

and  $u_1, u_2 \in X$  are the left-invariant vector fields on  $U(\mathcal{H})$ , generated by  $f_1$  and  $f_2$ , respectively, we have that

$$[u_1, u_2]_1 = [f_1, f_2] = \begin{bmatrix} \varphi_2^*\varphi_1 - \varphi_1^*\varphi_2 & 0 \\ 0 & \varphi_2\varphi_1^* - \varphi_1\varphi_2^* \end{bmatrix}$$

Let us denote the family of vector fields defined by the distribution  $X$ , also by  $X$  and suppose that the subspace  $V$  in the Hilbert space  $\mathcal{H}$  is finite-dimensional.

**Proposition 6** *The family of vector fields  $X$  on the manifold  $U(\mathcal{H})$  is completely nonholonomic.*

**Proof.** Let  $u_0$  and  $u_1$  be any two elements of  $U(\mathcal{H})$ . We have to show that there exists such control  $(t_1, X_1) \cdots (t_n, X_n)$ ,  $t_i \in \mathbb{R}$ ,  $X_i \in X$ , that  $\exp(t_1 X_1) \cdots \exp(t_n X_n) u_0 = u_1$ .

As the distribution  $X$  is left-invariant, we can assume that  $u_0 = 1$ . Consider the fiber bundle  $\pi : Isom(V, \mathcal{H}) \longrightarrow Gr_m(\mathcal{H})$ , where  $m = \dim(V)$ , and the connection form  $\mathcal{A} = \varphi^* d\varphi$  on it. As it follows from the theorem about the holonomy algebra of the connection  $\mathcal{A}$ , any two points  $\varphi_0, \varphi_1 \in Isom(V, \mathcal{H})$ , can be connected by a piece-wise smooth curve

$$\Phi : [0, 1] \longrightarrow Isom(V, \mathcal{H}), \quad \Phi(0) = \varphi_0, \Phi(1) = \varphi_1$$

Assume that  $\varphi_0$  and  $\varphi_1$  are such that  $\varphi_0(v) = v$  and  $\varphi_1(v) = u_1(v)$ ,  $\forall v \in V$ . Consider the fiber bundle  $\pi^\perp : Isom(V^\perp, \mathcal{H}) \longrightarrow Gr_m(\mathcal{H})$ , the fiber of which at a point  $W \in Gr_m(\mathcal{H})$  is  $Isom(V^\perp, W^\perp)$ . Consider two points  $\varphi_0^\perp, \varphi_1^\perp \in Isom(V^\perp, \mathcal{H})$ , such that  $\varphi_0^\perp(x) = x$  and  $\varphi_1^\perp(x) = u_1(x)$ ,  $\forall x \in V^\perp$ . It is clear that  $\varphi_0 \oplus \varphi_0^\perp = \mathbf{1}$  and  $\varphi_1 \oplus \varphi_1^\perp = u_1$ . On the manifold  $Gr_m(\mathcal{H})$  we have a curve  $\alpha : [0, 1] \longrightarrow Gr_m(\mathcal{H})$ ,  $\alpha = \pi\Phi$ , for which  $\alpha(0) = V$  and  $\alpha(1) = u_1(V)$ . Let  $\Phi^\perp : [0, 1] \longrightarrow Isom(V^\perp, \mathcal{H})$  be such lifting of the curve  $\alpha$  that  $\Phi^\perp(0) = \varphi_0^\perp$  and  $\Phi^\perp(1) = \varphi_1^\perp$ . It is clear that  $\{u(t) = (\Phi + \Phi^\perp)(t) \mid t \in [0, 1]\}$  is a piece-wise smooth family of unitary transformations of the Hilbert space  $\mathcal{H}$ , such that  $u(t)\varphi_0 = \Phi(t)$ ,  $u(0) = 1$ ,  $u(1) = u_1$ ,  $t \in [0, 1]$ . As the curve  $\Phi$  is horizontal, we obtain that  $\varphi_0^* u^* \dot{u} \varphi_0 = 0$ . Hence, we obtain that the piece-wise smooth curve  $u(t)$ ,  $t \in [0, 1]$ , connects the points  $u_0 = 1$  and  $u_1$  in  $U(\mathcal{H})$ , and at the same time is an integral curve of the dynamical polysystem  $X$  on the Lie group  $U(\mathcal{H})$ . ■

## 8 The Action of the Group $U(\mathcal{H})$ on the Principal Bundle $Isom(\mathbb{C}^m, \mathcal{H})$ . Schrödinger Equation and Berry Phase

Any element  $g$  of the unitary group  $U(\mathcal{H})$  acts on the total space  $Isom(\mathbb{C}^m, \mathcal{H})$  as  $\varphi \mapsto g\varphi$ , and on the base  $Gr_m(\mathcal{H})$  as  $X \mapsto g(X)$  (or, on the language of projectors:  $P \mapsto gPg^{-1}$ ). These two actions are correlated:

$$\pi(g\varphi) = g\pi(\varphi), \forall \varphi \in Isom(\mathbb{C}^m, \mathcal{H}) \text{ and } \forall g \in U(\mathcal{H})$$

It other words, the unitary transformation of the Hilbert space  $\mathcal{H}$  defines an automorphism of the principal bundle  $\pi : Isom(\mathbb{C}^m, \mathcal{H}) \longrightarrow Gr_m(\mathcal{H})$ . The connection form  $\mathcal{A} = \varphi^* d\varphi$  is invariant under the action of such automorphism

$$(g\varphi)^* d(g\varphi) = \varphi^* g^{-1} g d\varphi = \varphi^* d\varphi$$

For any element  $u$  of the Lie algebra  $\mathfrak{u}(\mathcal{H})$ , we have a one-parameter subgroup of the group  $U(\mathcal{H})$  generated by  $u$  —  $\{\exp(tu), t \in \mathbb{R}\}$ . The action of this group on the total space  $Isom(\mathbb{C}^m, \mathcal{H})$  defines a vector field  $\hat{u}$ , which can be described as  $\hat{u}_\varphi = u \circ \varphi$ ,  $\forall \varphi \in Isom(\mathbb{C}^m, \mathcal{H})$ . The action of this group on the base  $Gr_m(\mathcal{H})$ , defines a vector field  $\tilde{u}_P = [u, P]$ ,  $\forall P \in \mathcal{P}_m(\mathcal{H}) \cong Gr_m(\mathcal{H})$  which is the Hamiltonian vector field corresponding to the function  $\tilde{u} : Gr_m(\mathcal{H}) \longrightarrow \mathbb{R}$  defined as  $\tilde{u}(P) = \mathbf{i} \cdot \text{trace}(u^* \circ P)$  (see 23). The tangent vector  $\tilde{u}$  at a point  $X \in Gr_m(\mathcal{H})$ , can be described as  $\tilde{u}_X = u_{XX^\perp}$ , where  $u_{XX^\perp} \in Hom(X, X^\perp)$  is a component in the decomposition

$$u = \begin{bmatrix} u_{XX} & -u_{XX^\perp}^* \\ u_{XX^\perp} & u_{X^\perp X^\perp} \end{bmatrix} : \mathcal{H} = X \oplus X^\perp \longrightarrow \mathcal{H} = X \oplus X^\perp$$

As the actions of the group  $U(\mathcal{H})$  on the total space and the base are compatible, we have that  $\pi'(\hat{u}) = \tilde{u}$ .

**Remark 4** *This can be checked directly:*

$$\begin{aligned} \pi(\varphi) = \varphi\varphi^* &\Rightarrow d\pi = d\varphi\varphi^* + \varphi d\varphi^* \Rightarrow \\ \Rightarrow d\pi(\hat{u}) = u\varphi\varphi^* + \varphi(u\varphi)^* &= u \underbrace{\varphi\varphi^*}_{\pi(\varphi)} - \underbrace{\varphi\varphi^*}_{\pi(\varphi)} u = [u, \pi(\varphi)] \end{aligned}$$

As the action of the group  $U(\mathcal{H})$  preserves the connection form  $\mathcal{A}$ , we have that  $L_{\hat{u}}(\mathcal{A}) = 0$ , where  $L_{\hat{u}}$  denotes the Lie derivation by the vector field  $\hat{u}$ .

**Remark 5** *It can be checked directly:*

$$\begin{aligned} L_{\hat{u}}(\mathcal{A}) &= d(\mathcal{A}(\hat{u})) + (d\mathcal{A})(\hat{u}, \cdot) = \\ &= d(\varphi^* u \varphi) + (u \varphi)^* d\varphi - d\varphi^*(u \varphi) = d\varphi^* u \varphi + \varphi^* u d\varphi - \varphi^* u d\varphi - d\varphi^* u \varphi = 0 \end{aligned}$$

The vertical component of the vector field  $\hat{u}$  is measured by the connection form  $\mathcal{A}$  and is

$$\varphi \mathcal{A}(\hat{u}) = \varphi \varphi^* u \varphi = \pi(\varphi) u \varphi \quad (27)$$

Therefore, the horizontal component of the vector field  $\hat{u}$  at the point  $\varphi \in Isom(\mathbb{C}^m, \mathcal{H})$  is

$$u \varphi - \pi(\varphi) u \varphi = (1 - \pi(\varphi)) u \varphi \quad (28)$$

It is clear that in the expression 27, the term  $\pi(\varphi)$  is the projection operator on the subspace  $\text{Im}(\varphi) \subset \mathcal{H}$  and the term  $1 - \pi(\varphi)$ , in the expression 28, is the projection operator on the subspace  $\text{Im}(\varphi)^\perp \subset \mathcal{H}$ . Let us describe the vertical and horizontal components of the tangent vector  $u \varphi$  more explicitly. For any point  $\varphi \in Gr_m(\mathcal{H})$ , let

$$u = \begin{bmatrix} u_{11} & -u_{21}^* \\ u_{21} & u_{22} \end{bmatrix}$$

be the decomposition of the operator  $u \in U(\mathcal{H})$ , corresponding to the decomposition of the Hilbert space  $\mathcal{H} = \text{Im}(\varphi) \oplus \text{Im}(\varphi)^\perp$ . From the formulas 27 and 28, follows that the vertical component of the vector  $u \circ \varphi$  is

$$u_{11} \circ \varphi : \mathbb{C}^m \longrightarrow \text{Im}(\varphi)$$

and the horizontal component is

$$u_{21} \circ \varphi : \mathbb{C}^m \longrightarrow \text{Im}(\varphi)^\perp$$

Let  $H(t)$ ,  $t \in [0, T]$ ,  $T > 0$  be a time-dependent Hamiltonian, i.e., for any



$t \in [0, T]$ ,  $H(t)$  is an antisymmetric operator on the Hilbert space  $\mathcal{H}$ . This can be regarded as a time-dependent  $C^\infty$ -class function (Hamiltonian)

$$\tilde{H}(t) : Gr_m(\mathcal{H}) \cong \mathcal{P}_m(\mathcal{H}) \longrightarrow \mathbb{R}, \quad \tilde{H}(t)(P) = \mathbf{i} \cdot \text{trace}(H(t)^* \circ P)$$

which, together with the symplectic form (see the formula 22)  $\omega(X; \varphi, \psi) = i \cdot \text{trace}(\varphi^* \psi + \psi^* \varphi)$ , defines a time-dependent Hamiltonian vector field on the Grassmann manifold  $Gr_m(\mathcal{H})$ :  $\text{ham}(\tilde{H}(t))_P = [H(t), P]$ ,  $\forall P \in \mathcal{P}_m(\mathcal{H})$  (or equivalently:  $\text{ham}(\tilde{H}(t))_X = H(t)_{XX^\perp} \in \text{Hom}(X, X^\perp)$ ,  $\forall X \in Gr_m(\mathcal{H})$ ), and the corresponding time-dependent vector field  $H(t)$  on the total space  $\text{Isom}(\mathbb{C}^m, \mathcal{H})$ :  $\tilde{H}(t)_\varphi = H(t) \circ \varphi$ ,  $\forall \varphi \in \text{Isom}(\mathbb{C}^m, \mathcal{H})$ . The corresponding differential equation on the space  $\text{Isom}(\mathbb{C}^m, \mathcal{H})$  is

$$\dot{\varphi}(t) = H(t) \circ \varphi(t), \quad t \in [0, T] \quad (29)$$

and the differential equation (Hamiltonian (or Schrödinger) equation) on the Grassmann manifold is

$$\dot{P}(t) = [H(t), P(t)], \quad t \in [0, T] \quad (30)$$

Let  $P(t)$ ,  $t \in [0, T]$ ,  $P(0) \equiv P_0$  be a solution of the equation 30 and  $\varphi(t)$ ,  $t \in [0, T]$ ,  $\varphi(0) \cong \varphi_0 \in \pi^{-1}(P_0)$  be the corresponding solution of the equation 29, in the total space  $\text{Isom}(\mathbb{C}^m, \mathcal{H})$ . It is clear that  $\pi(\varphi(t)) = P(t)$ ,  $\forall t \in [0, T]$ , but the curve  $\varphi(t)$ , in general, is not the horizontal lifting of the curve  $P(t)$ , defined by the connection form  $\mathcal{A} = \varphi^* d\varphi$ , because, the tangent vector  $\dot{\varphi}(t)$  has the vertical component  $-H(t)_{11} \circ \varphi(t)$ , which, in general, is different from zero. This, pure geometric fact has a close relation with the effect, known in Physics as *Berry Phase*. The integral curve  $P(t)$ ,  $t \in [0, T]$  on the Grassmann manifold defines two isomorphisms between the fibers  $\pi^{-1}(P_0)$  and  $\pi^{-1}(P(T) \equiv P_1)$ :

1. For a point  $\sigma \in \pi^{-1}(P_0)$ , let  $\psi_\sigma(t)$ ,  $t \in [0, T]$  be the horizontal curve such that  $\psi_\sigma(0) = \sigma$  and  $\pi(\psi_\sigma(t)) = P(t)$ ,  $t \in [0, T]$ . Define an isomorphism  $\mathfrak{B}_1 : \pi^{-1}(P_0) \longrightarrow \pi^{-1}(P_1)$  as  $\mathfrak{B}_1(\sigma) = \psi_\sigma(T)$
2. For a point  $\sigma \in \pi^{-1}(P_0)$ , let  $\varphi_\sigma(t)$ ,  $t \in [0, T]$  be the solution of the equation 29, such that  $\varphi_\sigma(0) = \sigma$ . Define a mapping  $\mathfrak{B} : \pi^{-1}(P_0) \longrightarrow \pi^{-1}(P_1)$  as  $\mathfrak{B}(\sigma) = \varphi_\sigma(T)$ .

When the Hamiltonian curve  $P(t)$  on the Grassmann manifold is closed:  $P(0) = P(T)$ , then the isomorphism  $\mathfrak{B} : \pi^{-1}(P_0) \longrightarrow \pi^{-1}(P_0)$  is known as the *Berry phase* and the isomorphism  $\mathfrak{B}_1 : \pi^{-1}(P_0) \longrightarrow \pi^{-1}(P_0)$  is known as the *geometric Berry phase*.

We call a Hamiltonian curve  $P(t) \in Gr_m(\mathcal{H})$ ,  $t \in [0, T]$  *geometric* if the corresponding curve  $\varphi(t) \in Isom(\mathbb{C}^m, \mathcal{H})$ ,  $t \in [0, T]$ , in the total space, which is the solution of the equation 29 is horizontal; that is for each  $t \in [0, T]$  the tangent vector  $H(t) \circ \varphi(t)$  is horizontal. As the vertical component of this tangent vector is  $H(t)_{11} \circ \varphi$ , the curve  $P(t)$  is geometric if and only if  $H(t)_{11} = 0$  for each  $t \in [0, T]$ , where  $H(t)_{11} : X(t) \longrightarrow X(t)$  is the upper left component in the decomposition

$$H(t) = \begin{bmatrix} H(t)_{11} & -H(t)_{12}^* \\ H(t)_{12} & H(t)_{22} \end{bmatrix} : \mathcal{H} = X(t) \oplus X(t)^\perp \longrightarrow \mathcal{H} = X(t) \oplus X(t)^\perp$$

**Proposition 7** *For any curve  $Q : [0, T] \longrightarrow Gr_m(\mathcal{H})$  there exists such a time-dependent Hamiltonian  $H^Q(t) : \mathcal{H} \longrightarrow \mathcal{H}$ ,  $t \in [0, T]$  that the curve  $Q$  is a Hamiltonian curve for  $H^Q(t)$  and is geometric.*

**Proof.** The tangent vector  $\dot{Q}(t)$ , for any  $t \in [0, T]$  is an element of the tangent space of the Grassmannian:  $\dot{Q}(t) \in Hom(Q(t), Q(t)^\perp)$ . Consider the time-dependent Hamiltonian

$$H^Q(t) = \begin{bmatrix} 0 & -\dot{Q}(t)^* \\ \dot{Q}(t) & 0 \end{bmatrix}, \quad t \in [0, T]$$

it is clear that the curve  $Q : [0, T] \longrightarrow Gr_m(\mathcal{H}) \cong \mathcal{P}_m(\mathcal{H})$  satisfies the Hamiltonian (Schrödinger) equation  $\dot{Q}(t) = [H^Q(t), Q(t)]$ . This implies that the curve  $Q(t)$  is Hamiltonian. At the same time, for any  $t \in [0, T]$ , the component  $H_{11}^Q$  is automatically 0. Therefore, the tangent vectors  $H^Q \circ \varphi(t)$  for the points  $\varphi(t) \in \pi^{-1}(Q(t))$ ,  $t \in [0, T]$  are “pure horizontal”. ■

It is clear that for geometric curves on the base manifold, the two mappings  $\mathfrak{B}$  and  $\mathfrak{B}_1$  coincide. In other words, for geometric curves the **geometric Berry phase** and the **Berry phase** coincide.

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